Discrete Mathematics

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Course Contents

- Logic
- Sets and Set Operations
- Integers, Division and Matrices
- Relations
- Functions
- Sequences and summation
- Graphs
- Trees

Section 1.1: The Foundations: Logic

- **Mathematical Logic** is a tool for working with compound statements
- **Use of logic**
  - **In mathematics:**

    to prove theorems
  - **In computer science:**

    to prove that programs do what they are supposed to do

Section 1.1: Propositional Logic

Definition of a Proposition

- **Proposition** is a declarative statement **(that declares a fact)**
- **Proposition** is a statement that is either **True** (T) or **False** (F), but not both
- **Propositions** can be denoted by **Letters (p, q, r, ..)**
  - **True** value can be denoted by **T.**
  - **False** value can be denoted by **F.**

Note: Commands and questions are not propositions.

Examples of Propositions

- The following are **all** propositions:
  - "It is raining" (In a given situation)
  - "Amman is the capital of Jordan"
  - "1 + 2 = 3"
  - 1 + 1 = 4. (False) since, 1 + 1 = 2
- But, the following are **NOT** propositions:
  - "Who's there?" (Question)
  - "La la la la la." (Meaningless)
  - "Just do it!" (Command)
  - "1 + 2" (Expression with a non-true/false value)
  - "1 + 2 = *x*" (Expression with unknown value of *x*)
  - C++ is the best language (Opinion)

Examples of Propositions

Q: Are these propositions?

- 2+2=5
- Every integer is divisible by 12
- Microsoft is an excellent company

Examples of Propositions

- **Propositions** can be:
  - **Atomic:** consists of single proposition.
  - **Compound:** consists of one or more propositions connected by logical operators.

  - **P:** Today is Friday. : **F** **Atomic**
  - **Q:** 1 + 1 = 2. : **T** **Atomic**
  - **R:** $P \wedge Q$: **F** **Compound**

Some Popular Boolean Operators

Formal Name	**Nickname**	**Arity**	**Symbol**
Negation operator	NOT	Unary	$\neg$
Conjunction operator	AND	Binary	$\wedge$
Disjunction operator	OR	Binary	$\vee$
Exclusive-OR operator	XOR	Binary	$\oplus$
Implication operator	IMPLIES	Binary	$\rightarrow$
Biconditional operator	IFF	Binary	$\leftrightarrow$

The Negation Operator

Definition: Let p be a proposition then $\neg p$ is the **negation** of p (Not p , it is not the case that p).

EX: If P = "London is a city."

then $\neg p$ = "London is **not** a city" or " it is not the case that London is a city"

The **truth table** for NOT:

p	$\neg p$
F	T
T	F

Operand column

Result column

The Conjunction Operator

Definition: Let *p* and *q* be propositions, the proposition "*p* **AND** *q*" denoted by ($p \wedge q$) is called the **conjunction** of *p* and *q*.

Conjunction is True, if both P and q are true.

e.g. If *p* = "I will have salad for lunch" and

q = "I will have steak for dinner", then

$p \wedge q$ = "I will have salad for lunch **and** I will have steak for dinner"

Remember: "$\wedge$" points up like an "A", and it means "AND"

Conjunction Truth Table

- Note that a conjunction $p_1 \wedge p_2 \wedge \ldots \wedge p_n$ of n propositions will have 2^n rows in its truth table.

Operand columns		Result column
p	q	$p \wedge q$
F	F	F
F	T	F
T	F	F
T	T	T

"And", "But", "In addition to", "Moreover".

The Disjunction Operator

Definition: Let *p* and *q* be propositions, the proposition "*p* **OR** *q*" denoted by ($p \vee q$) is called the **disjunction** of *p* and *q*.

It is True, if any of P and q is true.

- **EX: Student who have taken calculus or computer science can take this class**

Disjunction Truth Table

- Note that $p \lor q$ means that *p* is true, or *q* is true, **or both** are true!
- So, this operation is also called **inclusive or**, because it **includes** the possibility that both *p* and *q* are true.

p	q	$p \lor q$
F	F	F
F	T	**T**
T	F	**T**
T	T	T

Note the differences from AND

Compound Statements

- Let p, q, r be simple statements
- We can form other compound statements, such as
  - $(p \vee q) \wedge r$
  - $p \vee (q \wedge r)$
  - $\neg p \vee \neg q$
  - $(p \vee q) \wedge (\neg r \vee s)$
  - and many others…

A Simple Exercise

Let p = "It rained last night",
q = "The sprinklers came on last night" ,
r = "The grass was wet this morning".

Translate each of the following into English:

$\neg p$ = **"It didn't rain last night"**

$r \wedge \neg p$ = **"The grass was wet this morning, and it didn't rain last night"**

$\neg r \vee p \vee q$ = **"Either the grass wasn't wet this morning, or it rained last night, or the sprinklers came on last night"**

The Exclusive Or Operator

The binary **exclusive-or** operator "⊕" (XOR) combines two propositions to form their logical "exclusive or" (exjunction?).

It is True, if <u>any</u> of P and Q is true, but not both.

e.g. p = "I will earn an A in this course"

q = "I will drop this course"

$p \oplus q$ = "I will either earn an A in this course, **or** I will drop it (but not both!)"

Exclusive-Or Truth Table

- Note that $p \oplus q$ means that p is true, or q is true, but **not both**!
- This operation is called **exclusive or**, because it **excludes** the possibility that both p and q are true.

p	q	$p \oplus q$
F	F	F
F	T	T
T	F	T
T	T	**F**

Note the difference from OR

Natural Language is Ambiguous

Note that English "or" can be ambiguous regarding the "both" case!

"Pat is a singer or
Pat is a writer" $\vee$

"Pat is a man or
Pat is a woman" $\oplus$

Need context to disambiguate the meaning!

For this class, assume "OR" means inclusive.

The Implication Operator

hypothesis conclusion

The **implication** $p \to q$ states that p implies q.

If p is true, then q is true; but if p is not true, then q could be either true or false.

e.g. Let p = "You get 100% on the final"
q = "You will get an A"

$p \to q$ = "If you get 100% on the final, then you will get an A"

Implication Truth Table

- $p \rightarrow q$ is **false** only when p is true but q is **not** true.

p	q	$p \rightarrow q$
F	F	T
F	T	T
T	F	**F**
T	T	T

The only False case!

The Implication Operator

- **$P \rightarrow Q$ has many forms in English Language:**
  - "P implies Q"
  - " If P, Q"
  - "If P, then Q"
  - "P only if Q“
  - "P is **sufficient** for Q"
  - "Q if P"
  - “Q when P"
  - “Q whenever P”
  - “Q follows from P”

Converse, Inverse, Contrapositive

Some terminology, for an implication $p \rightarrow q$:

- Its **converse** is: $q \rightarrow p$
- Its **inverse** is: $\neg p \rightarrow \neg q$
- Its **contrapositive** is: $\neg q \rightarrow \neg p$

Example of Converse, Inverse, Contrapositive

Write the converse, inverse and contrapositive of the statement "**if it is raining, then it is cloudy.**"

Converse	$Q \rightarrow P$	If it is cloudy, then it is raining
Contrapositive	$\neg Q \rightarrow \neg P$	If it is not cloudy, then it is not raining
Inverse	$\neg p \rightarrow \neg Q$	if it is not raining, then it is not cloudy

Note: The negation operation (¬) is different from the inverse operation.

Example of Converse, Inverse, Contrapositive

Proving the equivalence of p $\rightarrow$ q and its contrapositive using truth tables:

p	q	$\neg q$	$\neg p$	$p \rightarrow q$	$\neg q \rightarrow \neg p$
F	F	T	T	T	T
F	T	F	T	T	T
T	F	T	F	F	F
T	T	F	F	T	T

Biconditional ↔ Truth Table

- It is denoted by **P↔Q,** and it is read as **"P if and only if Q"**
- **It is true, if P and Q both have the same truth value.**
- Note this truth table is the exact **opposite** of ⊕'s!

  Thus, $P \leftrightarrow Q$ means $\neg(P \oplus Q)$

p	q	$p \leftrightarrow q$
F	F	T
F	T	F
T	F	F
T	T	T

In English:

- "*p* if and only if *q* "
- "If *p*, then *q*, and **conversely**"
- "*p* is **sufficient** and **necessary** for *q* "

Examples:

USING:

- **P:** John has a cat.
- **Q:** John has a dog.
- **R:** Today is sunny.
- **S:** it rains.
- **T:** I wear my coat.

WE CAN BUILD:

- **~R:** Today is not sunny.
- **P $\wedge$ Q**: John has a cat and a dog.
- **P $\vee$ Q**: John has a cat or a dog.
- **P $\oplus$ Q**: John has a pet; it is either a cat or a dog.
- **S $\rightarrow$ T:** If it rains, I will wear my coat.
- **S $\leftrightarrow$ T**: If it rains, I will wear my coat, and conversely

Truth Tables of Compound Proposition:

- **<u>Note:</u>** If a compound proposition has **n** distinct simple components, then it will have $\mathbf{2^n}$ rows in its truth table, as this is the number of possible combinations of n components, each with 2 possible truth values **T** or **F**.
- Precedence of Logical Operators

Operator	**Precedence**
( )	1
$\neg$	2
$\wedge$, $\vee$	3
$\rightarrow$, $\leftrightarrow$	4
Left to Right	5

Example: Truth Table of $(p \vee q) \wedge r$

p	q	r	$p \vee q$	$(p \vee q) \wedge r$
F	F	F	F	F
F	F	T	F	F
F	T	F	T	F
F	T	T	T	T
T	F	F	T	F
T	F	T	T	T
T	T	F	T	F
T	T	T	T	T

Examples:

- **Example:** The following truth table is used to represent the compound proposition: $(P \wedge Q) \vee (\sim P)$

P	**Q**	$P \wedge Q$	$\sim P$	$(P \wedge Q) \vee (\sim P)$
T	T	T	F	T
T	F	F	F	F
F	T	F	T	T
F	F	F	T	T

Translation English Sentences into Logical Expressions

- If you are a **c**omputer science major **or** you are not a **f**reshman, then you can **a**ccess the internet from campus :

  is translated to: $(c \vee \neg f) \rightarrow a$

- You got an A in this class, **but** you did not do every exercise in the book.

  is translated to: P $\wedge$ $\neg$ Q.

Translation English Sentences into Logical Expressions

- **if** it is hot outside buy an ice cream, **and if** you buy an ice cream it is hot outside.

is translated to: $P \rightarrow Q \wedge Q \rightarrow P \quad \equiv \quad P \leftrightarrow Q$

- **You can't drive a car if you are a student unless you are older than 18 years old.**

is translated to: $(Q \wedge \neg R) \rightarrow \neg P$

Logic and Bit Operations

- **Bit** has two values: 0, 1

Truth value	Bit
F	0
T	1

- **Boolean Variable:** a variable that is either true or false.
- **- Bit operation** corresponds to logical connectives:

Logical Operator	Bit operator
$\neg$	NOT
$\wedge$	AND
$\vee$	OR
$\oplus$	XOR

Logic and Bit Operations

- **- Bit string:** it is a sequence of zero or more bits.
- - **String Length:** number of bits in the Bit string.
- Find the bitwise AND, bitwise OR, and bitwise XOR of the bit strings 0110110110 and 1100011101.

	0110110110
	1100011101
Bitwise AND	0100010100
Bitwise OR	1110111111
Bitwise XOR	1010101011

Section 1.2: Propositional Equivalences:

- **Logical equivalence:**

  Compound propositions that have the same truth values in all possible cases are called logically equivalent.

Logical Equivalence

- $\neg p \vee q$ is **logically equivalent** to $p \rightarrow q$

p	q	$\neg p \vee q$	$p \rightarrow q$
F	F	T	T
F	T	T	T
T	F	F	F
T	T	T	T

Proving Equivalence via Truth Tables

Example: Prove that $p \vee q$ and $\neg(\neg p \wedge \neg q)$ are logically equivalent.

p	q	$p \vee q$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$\neg(\neg p \wedge \neg q)$
F	F	F	T	T	T	F
F	T	T	T	F	F	T
T	F	T	F	T	F	T
T	T	T	F	F	F	T

Propositional Equivalence, Tautologies and Contradictions

- A **tautology** is a compound proposition that is always **true.**

  e.g. $p \vee \neg p \equiv \mathbf{T}$

- A **contradiction** is a compound proposition that is always **false.**

  e.g. $p \wedge \neg p \equiv \mathbf{F}$

- Other compound propositions are **contingencies**.

  e.g. $p \rightarrow q$, $p \vee q$

Tautology

Example: $p \rightarrow p \vee q$

p	q	$p \vee q$	$p \rightarrow p \vee q$
F	F	F	T
F	T	T	T
T	F	T	T
T	T	T	T

Equivalence Laws $\Leftrightarrow$

- Identity: $p \wedge \mathbf{T} \Leftrightarrow p$, $p \vee \mathbf{F} \Leftrightarrow p$
- Domination: $p \vee \mathbf{T} \Leftrightarrow \mathbf{T}$, $p \wedge \mathbf{F} \Leftrightarrow \mathbf{F}$
- Idempotent: $p \vee p \Leftrightarrow p$, $p \wedge p \Leftrightarrow p$

- Double negation: $\neg\neg p \Leftrightarrow p$
- Commutative: $p \vee q \Leftrightarrow q \vee p$, $p \wedge q \Leftrightarrow q \wedge p$

- Associative: $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$
  $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$

More Equivalence Laws

- Distributive: $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$
  $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$
- De Morgan's:
  $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$
  $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$
- Trivial tautology/contradiction:
  $p \vee \neg p \Leftrightarrow \mathbf{T}$, $p \wedge \neg p \Leftrightarrow \mathbf{F}$
- $\neg(p \rightarrow q) \Leftrightarrow \neg(\neg p \vee q) \Leftrightarrow p \wedge \neg q$

Augustus
De Morgan
(1806-1871)

Implications / Biconditional Rules

1. $p \rightarrow q \equiv \neg p \vee q$
2. $\neg (p \rightarrow q) \equiv p \wedge \neg q$
3. $p \rightarrow q \equiv \neg q \rightarrow \neg p$ (contrapositive)
4. $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
5. $\neg (p \leftrightarrow q) \equiv p \oplus q$

Proving Equivalence using Logic Laws

- Example 1. show that $\neg (P \rightarrow Q)$ and $P \land \neg Q$ are logically equivalent

$(P \rightarrow Q) \equiv \neg(\neg P \lor Q)$

$\equiv \neg \neg P \land \neg Q$

$\equiv P \land \neg Q$

Proving Equivalence using Logic Laws

- Example 2. show that $(P \wedge Q) \rightarrow (P \vee Q)$ is a tautology

$\neg (P \wedge Q) \vee (P \vee Q)$	Implication rule
$(\neg P \vee \neg Q) \vee (P \vee Q)$	De morgan Law
$(\neg P \vee P) \vee (\neg Q \vee Q)$	Associative and commutative
$T \vee T$	Negation law
T	

Proving Equivalence using Logic Laws

Example 3. Show that $\neg (P \vee (\neg P \wedge Q))$ and $(\neg P \wedge \neg Q)$ are logically equivalent.

$\neg \ (P \vee (\neg P \wedge Q))$

$\equiv \neg P \wedge \neg (\neg P \wedge Q)$ De Morgan

$\equiv \neg P \wedge (\neg(\neg P) \vee \neg Q)$ De Morgan

$\equiv \neg P \wedge (P \vee \neg Q)$ Double negation

$\equiv (\neg P \wedge P) \vee (\neg P \wedge \neg Q)$ Distributive

$\equiv \mathbf{F} \vee (\neg P \wedge \neg Q)$ Negation

$\equiv (\neg P \wedge \neg Q)$ Identity

Proving Equivalence using Logic Laws

Example 4: Show that $\neg (\neg (P \to Q) \to \neg Q)$ is a contradiction.

$\neg (\neg (P \to Q) \to \neg Q)$

$\equiv \neg (\neg (\neg P \lor Q) \to \neg Q)$ Equivalence

$\equiv \neg ((P \land \neg Q) \to \neg Q)$ De Morgan

$\equiv \neg (\neg (P \land \neg Q) \lor \neg Q)$ Equivalence

$\equiv \neg (\neg P \lor Q \lor \neg Q)$ De Morgan

$\equiv \neg (\neg P \lor \mathbf{T})$ Trivial Tautology

$\equiv \neg (\mathbf{T})$ Domination

$\equiv \mathbf{F}$ Contradiction

Section 1.3: Predicates and Quantifiers

- **Predicates: Statement involving variable**, such as

  $x > 3$

  $x = y + 3$

  $x+y=z$

- "x is greater than 3" has two parts

  First part: x , is a variable.

  Second part: "is greater than 3", is a predicate.

  "x is greater than 3" can be denoted by the **propositional function** $P(x)$.

  $P(x)$: $x > 3$, let $x = 4$, then $P(4)$ is true,

  let $x = 1$, then $P(1)$ is false.

Section 1.3: Predicates and Quantifiers

- The statement P(x) is said to be the value of the propositional function P at x.
- Once a value is assigned to x , P(x) becomes a proposition and has a truth value.
- Convention: Lowercase variables x, y, z... denote objects, uppercase variables P, Q, R… denote propositional functions (predicates).
- In general , a statement involving the n variable x1,x2,x3, …………, xn can be denoted by P(x1,x2,x3……..nx).

Section 1.3: Predicates and Quantifiers

- Ex: Q(x, y): x= y+3

  Q(3,0): 3 = 0 + 3 T

- Ex: R(X,Y,Z): x+y = z

  R(1,2,3): 1+2=3 T

- Ex: P(x) = “x is a prime number”,

  P(3) is the proposition “3 is a prime number.” T

Quantifiers

- Quantification expresses the extent to which a predicate is true over a range of elements.
- We will focus on two types of quantification:

Quantification → Universal Quantification **(∀), all**

→ Existential Quantification **(∃), some**

Universal Quantification

- Universal quantification: Tells us that a predicate of P(x) is true for every element a particular domain, called **Domain** (D) (or the **Universes of Discourse** (U.D))
- The notation $\forall x\ P(x)$ denotes the universal quantification of P(x). It is read as: "for all x P(x)" or "for every x P(x)"

  $\forall$: is called the universal quantifier
- For example:
  - *For every* triangle *T*, the sum of the angles of *T* is 180 degrees.

Universal Quantification

- <u>In general</u>: when the elements of UD are x1,x2,x3,…,xn. It follows that $\forall x\, P(x)$ is the conjunction of: $P(x1) \wedge P(x2) \wedge \ldots P(xn)$
- Note: Specifying the UD is important when quantifiers are used.

Let $P(x)$: $x^2 \geq x$, Domain is the set {0.5, 1, 2, 3}.

- $\forall x\, P(x) \equiv P(0.5) \wedge P(1) \wedge P(2) \wedge P(3)$

  $\equiv F \wedge T \wedge T \wedge T$

  $\equiv F$

Universal Quantification

- Example: What is the truth value of

$$\forall x\ (x^2 \geq x)\ .$$

- If UD is all real numbers, the truth value is false (take x = 0.5, this is called a **counterexample**).

- If UD is the set of integers, the truth value is true.

Example

Suppose that $P(x)$ is the statement "$x + 3 = 4x$" where the domain is the set of integers. Determine the truth values of $\forall x\, P(x)$. Justify your answer.

It is clear that $P(1)$ is True, but $P(x)$ is False for every $x \neq 1$ (take $x = 2$ as a counterexample). Thus, $\forall x\, P(x)$ is **False.**

Existential Quantification

- Existential Quantification: Tells us that there is one or more element under consideration for which the predicate is true
- $\exists\ x\ Q(x)$: There exists an element x in the universe of discourse such that $Q(x)$ is true.

In general: when the elements of UD are x1, x2, x3,…, x_n. It follows that $\exists$ x P(x) is the disjunction of: P(x1) $\vee$ P(x2) $\vee$ … P(xn)

Let $P(x)$: $x^2 \geq x$, Domain is the set {0.5, 1, 2, 3}.

$$\begin{aligned} \exists x\, P(x) &\equiv P(0.5) \vee P(1) \vee P(2) \vee P(3) \\ &\equiv F \vee T \vee T \vee T \\ &\equiv T \end{aligned}$$

Existential Quantification

- Example 1: Let $Q(x)$: $x = x + 1$, Domain is the set of all real numbers:

  - The truth value of $\exists x\, Q(x)$ is false (as there is no real x such that $x = x + 1$).

- Example 2: Let Q(x): x2 = x, Domain is the set of all real numbers:

  - The truth value of $\exists x\, Q(x)$ is true (take $x = 1$).

Summary

- In order to prove the quantified statement $\forall x\, P(x)$ is true
  - It is **not** enough to show that $P(x)$ is true for some $x \in D$
  - You must show that $P(x)$ is true for every $x \in D$
  - You can show that $\exists x \neg P(x)$ is false

- In order to prove the universal quantified statement $\forall x\ P(x)$ is false
  - It is enough to exhibit some $x \in D$ for which $P(x)$ is false
  - This x is called the **counterexample** to the statement $\forall x\ P(x)$ is true

Summary

- In order to prove the existential quantified statement $\exists\ x\ Q(x)$ is true
  - It is enough to exhibit some $x \in D$ for which $Q(x)$ is true

- In order to prove the existential quantified statement $\exists\ x\ Q(x)$ is false
  - It is **not** enough to show that $Q(x)$ is false for some $x \in D$
  - You must show that $Q(x)$ is false for every $x \in D$

ORDER OF QUANTIFIER

Statement	When true	When false
∀x∀y P(x, y) ∀y∀x P(x, y)	P(x, y) is true for every pair x, y.	There is a pair x, y for which P(x, y) is false
∀x∃y P(x, y)	For every x, there is a y for which P(x, y) is true	There is x, such that P(x, y) is false
∃x∀y P(x, y)	There is x for which P(x, y) is true for every y.	For every x there is a y for which P(x, y) is false
∃x∃y P(x, y) ∃y∃x P(x, y)	There is a pair x, y for which P(x, y) is true	P(x, y) is false for every pair x, y.

Example

Suppose that the universe of discourse of $P(x, y)$ is $\{1, 2, 3\}$. Write out the following propositions using disjunctions and conjunctions:

$\exists x\, P(x, 2)$	**$P(1,2) \lor P(2, 2) \lor P(3, 2)$**
$\forall y\, P(3, y)$	**$P(3,1) \land P(3,2) \land P(3,3)$**
$\forall x\, \forall y\, P(x, y)$	**$P(1,1) \land P(1,2) \land P(1,3) \land P(2,1) \land P(2,2) \land P(2,3) \land P(3,1) \land P(3,2) \land P(3,3)$**
$\exists x\, \exists y\, P(x, y)$	**$P(1,1) \lor P(1,2) \lor P(1,3) \lor P(2,1) \lor P(2,2) \lor P(2,3) \lor P(3,1) \lor P(3,2) \lor P(3,3)$**
$\exists x\, \forall y\, P(x, y)$	**$(P(1,1) \land P(1,2) \land P(1,3)) \lor (P(2,1) \land P(2,2) \land P(2,3)) \lor (P(3,1) \land P(3,2) \land P(3,3))$**
$\forall x\, \exists y\, P(x, y)$	**$(P(1,1) \lor P(1,2) \lor P(1,3)) \land (P(2,1) \lor P(2,2) \lor P(2,3)) \land (P(3,1) \lor P(3,2) \lor P(3,3))$**

Nesting of Quantifiers

- **Example:** UD: all real numbers
- $\forall x\ \forall y\ (x+y = 0)$ false
- $\forall \mathbf{x}\ \exists \mathbf{y}\ (x+y = 0)$ true (inverse)
- $\exists\ \mathbf{x}\ \exists \mathbf{y}\ (x+y = 0)$ true
- $\exists \mathbf{x}\ \forall \mathbf{y}\ (x+y = 0)$ false

Precedence of Quantifiers

- **Precedence of Quantifiers:**
- The quantifiers $\forall$ and $\exists$ have higher precedence than all logical operators from propositional calculus.
- For example, $\forall x P(x) \lor Q(x)$ is the disjunction of $\forall x P(x)$ and $Q(x)$.
- In other words, it means $(\forall x P(x)) \lor Q(x)$ rather than $\forall x (P(x) \lor Q(x))$.

Translation English Sentences into Logical Expressions

- EX1:“Every student in this class has studied math and C++ course”.

<u>UD is the students in this class:</u>

Translated to: $\forall x\ (M(x) \land CPP(x))$

<u>But if the UD is all people:</u>

“For every person *x*, if *x* is a student in this class then *x* has studied math and C++”

Translated to: $\forall x\ (S(x) \rightarrow M(x) \land CPP(x))$

Translation English Sentences into Logical Expressions

- EX2: "Some student in this class has studied math and C++ course".

UD is the students in this class:

Translated to: $\exists x\ (M(x) \land CPP(x))$

But if the UD is all people:

Translated to: $\exists x\ (S(x) \land M(x) \land CPP(x))$

Translation English Sentences into Logical Expressions

- EX3: “Some student in this class has visited Aqaba”

  UD: student in this class . It means:

  “There is a student x in this class who visited Aqaba”

  Translated to: $\exists\, x\, A(x)$

- But if the UD: all people

  There is a person x having the properties that x is a student in this class and x has visited Aqaba”

  $\exists\, x\, (S(x) \wedge A(x))$

Translation English Sentences into Logical Expressions

- EX4: No one is perfect

  $\forall x \neg P(x)$

- EX5: All your friends are perfect.

  F(x): your friend

  P(x): perfect

  $\forall x (F(x) \rightarrow P(x))$

Translation English Sentences into Logical Expressions

- EX6: Let P(x) be the statement “x can speak French” and Q(x) be the statement “x knows C++”. The domain is all students in the school. Express the following statement using quantifiers and logical operator:

A. No student at your school can speak French or knows c++.

$\forall x \neg (P(x) \vee Q(x))$

B. There is a student at your school who can speak French but does not know C++.

$\exists x (P(x) \wedge \neg Q(x))$

If **Domain: all Students**

$\exists x (S(x) \wedge P(x) \wedge \neg Q(x))$

Translation English Sentences into Logical Expressions

- Everybody likes somebody."
  - For everybody, there is somebody they like,
    - **$\forall$x $\exists$y Likes(x,y)**
  - or, there is somebody (a popular person) whom everyone likes?
    - **$\exists$y $\forall$x Likes(x,y)**

Translation English Sentences into Logical Expressions

- **Exercise: express the statement:**

"if a person is a female and is a parent, then this person is someone's mother"

F(x): person is a female

P(x): person is a parent

M(x, y): x is the mother of y

- **Solution:**

$$\forall x \, \exists y \, ((F(x) \land P(x)) \rightarrow M(x, y))$$

Translation English Sentences into Logical Expressions

Translate the statement "The sum of two positive integers is always positive" into a logical expression.

"For every two integers, if these integers are both positive, then the sum of these integers is positive."

$\forall x \forall y((x > 0) \wedge (y > 0) \rightarrow (x + y > 0))$,

"The sum of two positive integers is always positive"

$$\forall x \forall y(x + y > 0)$$

Where the domain for both variables consists of all positive integers.

Translation English Sentences into Logical Expressions

Translate the statement "Every real number except zero has a multiplicative inverse." (A **multiplicative inverse** of a real number x is a real number y such that xy = 1.)

Solution:

We first rewrite this as "For every real number x except zero, x has a multiplicative inverse." We can rewrite this as "For every real number x, if $x \neq 0$, then there exists a real number y such that $xy = 1$." This can be rewritten as

$$\forall x((x \neq 0) \rightarrow \exists y(xy = 1)).$$

Negations

- $\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$
- $\neg \exists x\, Q(x) \equiv \forall x\, \neg Q(x)$

- Ex1:All Americans eat cheeseburgers -- $\forall x\, P(x)$

  Negation: There is an American who does not eat cheeseburgers $\exists x \neg P(x)$
- Ex2: Every student in the class has taken calculus. $\forall x\, P(x)$

  There is a student in the class who has not taken Calculus. $\neg \forall x\, P(x) \equiv \exists x \neg p(x)$

Negations

- Ex3:There is a student in the class who has taken Calculus $\exists x\ p(x)$

  Every student in the class has not taken calculus.

  $\neg \exists x\ P(x) \equiv \forall x \neg p(x)$

Ex4: what are the negations of the following statements?

A. $\forall x\ (x*x > x)$

Sol: $\neg \forall x\ (x*x > x) \rightarrow \exists x \neg (x*x > x) \rightarrow \exists x\ (x*x \leq x)$

B. $\exists x\ (x*x = 2)$

Sol : $\neg \exists x\ (x*x = 2) \rightarrow \forall x \neg (x*x = 2) \rightarrow \forall x\ (x*x \neq 2$

Negations

- Example: Show that $\neg\forall x(P(x) \rightarrow Q(x))$ and $\exists x(P(x)\wedge\neg Q(x))$ are logically equivalent.

  $\neg\forall x(P(x) \rightarrow Q(x))$

  $\exists x(\neg(P(x) \rightarrow Q(x)))$

  $\exists x(P(x)\wedge\neg Q(x))$

- Example: Let $P(x)$ is the statement "$x^2 - 1 = 0$", where the domain is the set of real numbers R.

  - The truth value of $\forall x P(x)$ is **False**

  - The truth value of $\exists x P(x)$ is **True**

  $\neg \forall x P(x) \equiv \exists x (x 2 - 1 \neq 0)$, which is **True**

  $\neg \exists x P(x) \equiv \forall x (x 2 - 1 \neq 0)$, which is **False**